

Density functional study of a ferromagnetic ferroelectric $\text{LaMnO}_3/\text{BaTiO}_3$ superlattice

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Using the GGA+ U density functional, we have calculated the lattice constants, atomic positions, magnetization, and ferroelectric polarization of a $(\text{LaMnO}_3)_{4.5}(\text{BaTiO}_3)_{4.5}$ superlattice containing five LaO and TiO_2 planes and four MnO_2 and BaO planes. Although LaMnO_3 is antiferromagnetic, it is ferromagnetic in the superlattice. An approximation to the ferroelectric polarization, obtained from a comparison of superlattice and bulk crystal atomic displacements and unit cell volumes, is found to agree reasonably well with that obtained from a Berry phase calculation. The electric polarization is polarized along the longer in-plane lattice vector, while the Mn spins, when the spin-orbit interaction and spin noncollinearity are included, have different magnitudes, but all point in essentially the same direction as the electric polarization.

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I. INTRODUCTION

Multiferroic materials form a class of compounds, which have recently received a lot of attention from the material physics community. In order for a compound to belong to this group, it requires at least two order parameters to coexist in the same phase. Considering elastic, electric, and magnetic properties, there are three distinct groups of multiferroics: piezoelectrics, which have elastic and electric properties, piezomagnets, which have elastic and magnetic properties, and magnetoelectrics, where ferroelectric (or antiferroelectric) and magnetic properties coexist. If the first two groups of multiferroics are relatively common, the magnetoelectric materials are much less so. A very important group of magnetoelectric multiferroics is formed by hexagonal rare-earth manganites, which have the overall formula RMnO_3 ($R = \text{Y, Ho, Er, Tm, Yb, Lu, Sc}$) and are ferroelectric, but all are antiferromagnetic.¹⁻⁴

Ferroelectric ferromagnets are extremely rare. One compound, which has received a lot of attention recently, is BiMnO_3 . It is well established that BiMnO_3 is a magnetic material with a magnetization of about $3.9 \mu_B/\text{f.u.}$, a fact supported by a number of experimental and theoretical papers,⁵⁻⁷ and it had also been believed to be ferroelectric with the noncentrosymmetric $C2$ space group, a conclusion based on powder neutron diffraction experiments.^{8,9} Seshadri and Hill¹⁰ attributed this structure to the stereochemical activity of the Bi $6s^2$ lone pair. However, very recently, Belik *et al.*,¹¹ having used selected area electron diffraction, convergent beam electron diffraction, and Rietveld analysis of neutron diffraction data on polycrystalline samples, concluded that BiMnO_3 crystallizes in the centrosymmetric $C2/c$ space group. Having first determined that the actual structure depends sensitively on the oxygen stoichiometry,¹² Montanari *et al.*¹³ used neutron studies on polycrystalline BiMnO_3 as a function of temperature and magnetic field to conclude that the space group is $C2/c$. Baettig *et al.*¹⁴ performed full computational structural optimizations of BiMnO_3 using the LDA+ U method, finding that although the two proposed^{8,9} polar $C2$ structures existed, they are higher in energy than an antiferroelectric centrosymmetric $C2/c$ structure.

As far as we know, there are only two ferromagnetic ferroelectrics thus far synthesized (or three if one counts

BiFeO_3 , an antiferromagnetic ferroelectric with a small spin canting, resulting in a negligible ferromagnetic moment, and which in a 70 nm thick film was reported¹⁵ to have a saturation magnetization of $\sim 1 \mu_B$). One of these is $\text{Bi}_2\text{NiMnO}_6$. This compound was recently synthesized by Azuma *et al.*¹⁶ as a heavily distorted double perovskite with 4 f.u. in a monoclinic unit cell. We¹⁷ recently reported the results of generalized gradient approximation (GGA) and GGA+ U calculations of $\text{Bi}_2\text{NiMnO}_6$ in which the effects of spin orbit and spin noncollinearity were included. A magnetization of $4.94 \mu_B/\text{f.u.}$ (4.99 with U) and a polarization of $18.41 \mu\text{C}/\text{cm}^2$ (18.18 with U) were obtained. The other compound of which we are aware was synthesized by Gajek *et al.*¹⁸ They reported that $\text{La}_{0.1}\text{Bi}_{0.9}\text{MnO}_3$ is a ferroelectromagnet, which retains its properties down to a 2 nm thick film, and have used it as a spin filtering tunnel barrier, the magnetization and electric polarization of which can be switched independently, giving rise to four different resistance states. The extraordinary challenge of creating these ferroelectromagnetic compounds suggests a superlattice approach. Murugavel *et al.*¹⁹ have created a $(\text{PCMO})_{10}(\text{BST})_9$ superlattice, where PCMO is ferromagnetic $\text{Pr}_{0.85}\text{Ca}_{0.15}\text{MnO}_3$ and BST is ferroelectric $\text{Ba}_{0.6}\text{Sr}_{0.4}\text{TiO}_3$. The superlattice had ferromagnetic and ferroelectric transition temperatures, $T_C = 120 \text{ K}$ and $T_E = 140 \text{ K}$, and showed a 40% magnetoresistance in a 7 T field at 80 K. Singh *et al.*²⁰ studied the structure of $(\text{La}_{0.7}\text{Ca}_{0.3}\text{MnO}_3)_n/(\text{BaTiO}_3)_m$ superlattices with various values of n and m and measured magnetizations, ferroelectric polarizations, and Curie temperatures. Interestingly, they found that a superlattice with $n=m=4$ was ferromagnetic up to 400 K, while thick films had a Curie temperature of only 145 K. The Curie temperature of bulk $\text{La}_{0.7}\text{Ca}_{0.3}\text{MnO}_3$ is about 250 K.

There are only two calculations of which we are aware that involve a ferroelectric interfacing with a ferromagnet. One involves a Fe/BaTiO_3 superlattice. The authors²¹ fixed the transverse (i.e., in the interface plane) lattice constant of the superlattice²² at the experimental BaTiO_3 lattice constant. Because this is smaller than their calculated BaTiO_3 lattice constant, ferroelectric polarization in the longitudinal direction was assured. They found a $0.37 \mu_B$ difference between the sum of the Ti and Fe magnetic moments in the top and

bottom interfaces (where “top” is the direction of BaTiO₃ polarization) and suggested that if the magnetization of the surface of a few layers of iron on BaTiO₃ could be controlled by changing the sign of the ferroelectric polarization, useful applications might be found. The centered 2×2 Fe lattice constant is larger than the BaTiO₃ lattice constant to which it is nearly matched, so in actual fact, the transverse lattice constant of the superlattice will be slightly larger than that of BaTiO₃ and the ferroelectric polarization would be expected to lie in the plane in this case, but the concept is, nevertheless, interesting. The other calculation,²³ for a Co₂MnSi/BaTiO₃ superlattice, calculated lattice constants and found the Ti to be polarized at the Co₂/TiO₂ interface but probably not at the MnSi/TiO₂ interface.

In this paper, we present the results of completely relaxed electronic structure calculations for a LaMnO₃/BaTiO₃ superlattice, similar to that described in Ref. 20 except for the replacement of La_{0.7}Ca_{0.3}MnO₃ with LaMnO₃. The only reason for replacing La_{0.7}Ca_{0.3}MnO₃ with LaMnO₃ was to keep the calculation of feasible size, even if, by doing so, we expected to replace the ferromagnetism of Ca doped LaMnO₃ with the antiferromagnetism of pure LaMnO₃. In the following section, we describe our computational method, and in the last section, we present and discuss our results.

II. COMPUTATIONAL DETAILS

Our calculations were performed using the projected augmented wave²⁴ method, as implemented in the VASP code.²⁵ The generalized gradient approximation²⁶ plus Hubbard U approximation²⁷ (GGA+ U) was used for the exchange-correlation energy density functional. In order to properly describe LaMnO₃ as a semiconductor rather than a metal, we had to use the fairly large value of 7.5 eV for the Mn U - J parameter,²⁸ equal to the value obtained from $2p$ core-level photoemission spectra²⁹ and less than the 9.22 eV obtained from “constrained” density functional calculations.³⁰ For Ti, we assumed that U - J =0. The Brillouin zone was sampled with a $7 \times 7 \times 3$ \mathbf{k} -point grid. The basis set used contained all plane waves up to 20.79 Ry. The atomic positions were adjusted until all the forces on them were less than 10 meV/Å and the VASP code adjusted the lattice constants using a stress criterion based on the force criterion.

We originally constructed our superlattice to match that measured in Ref. 20, with four layers of BaTiO₃ alternating with four³¹ of LaMnO₃. However, we were unable to get the forces to converge. Because the forces at the MnO₂/BaO interface were much larger than those at the LaO/TiO₂ interface, we constructed a superlattice with 4.5 layers of each compound, which contained only LaO/TiO₂ interfaces and had no further convergence problems. This seems to confirm what was found in Ref. 21 and assumed to be true in Ref. 23, namely, that the TiO₂ interface with a ferromagnet is much more stable than the BaO interface with the ferromagnet. The large electric fields which can occur between dissimilar interfaces may have played a role in the failure to converge. With the lattice constants and atomic positions thus obtained, the spin-orbit interaction and a continuously variable spin

TABLE I. Calculated lattice constants in Å and unit cell volumes in Å³ of BaTiO₃ and a double BaTiO₃ unit cell for comparison with the LaMnO₃ unit cell and with the experiment. The last row lists the (BaTiO₃)_{4.5}(LaMnO₃)_{4.5} lattice constants and unit cell volume.

| | a | b | c | Vol. |
|--------------------|--------|--------|---------|-----------|
| BaTiO ₃ | 3.9325 | 3.9325 | 3.9567 | 61.1886 |
| | 5.5614 | 5.5614 | 7.9134 | 244.7544 |
| Expt. ^a | 3.992 | 3.992 | 4.036 | 64.317954 |
| LaMnO ₃ | 5.5612 | 5.7607 | 7.8083 | 250.1240 |
| Expt. ^b | 5.532 | 5.742 | 7.669 | 243.569 |
| Supercell | 5.6498 | 5.7181 | 35.8470 | 1158.0591 |

^aReference 32.

^bReference 33.

quantization direction were included, and the calculation again brought to self-consistency.

III. RESULTS AND DISCUSSION

The calculated lattice constants of BaTiO₃ (Ref. 32) and LaMnO₃ (Ref. 33) are compared with the experiment in Table I. Also listed are lattice constants for four formula units of BaTiO₃ for comparison with the LaMnO₃ unit cell. The discrepancies between theory and experiment are typical for density functional calculations, although the sign of the discrepancy in BaTiO₃ is unexpected from a GGA density functional calculation. LaMnO₃ is triclinic but very close to being orthorhombic. This can be demonstrated by observing the ratio of the unit cell volume to the product of the lattice constants, 250.124/250.150 (calculated) and 243.569/243.592 (experiment). The antiferromagnetic LaMnO₃ magnetization was calculated to be $4.226 \mu_B$ per Mn in fairly poor agreement with the experimental value³⁴ of $3.7 \pm 0.1 \mu_B$. Better agreement could be obtained with smaller values of U - J . For example, $3.827 \mu_B$ was obtained with U - J =3.5 eV. It is unfortunate that no single value of the U - J parameter can yield both the correct energy gap and the correct magnetization. Using the Berry phase method,³⁵ the BaTiO₃ ferroelectric polarization was found to be $27.05 \mu\text{C}/\text{cm}^2$, in good agreement with the experimental value³⁶ of $26.0 \mu\text{C}/\text{cm}^2$. Berry phases exist only for filled bands, which caused us to choose the larger value of U - J .

The superlattice calculation (with no spin orbit or noncollinearity) was initiated with an antiferromagnetic Mn spin density similar to that of bulk LaMnO₃, but it converged to a ferromagnetic distribution. One might suspect that an antiferromagnetic local energy minimum exists, but if it does, VASP manages to avoid it. The last row of Table I lists the superlattice lattice constants. Note that the a and b constants are greater than the average of the BaTiO₃ and LaMnO₃ a and b . Similarly, c and the unit cell volume are larger than 2.25 times the sum of BaTiO₃ and LaMnO₃ c 's and volumes. There is at least one and perhaps two independent causes for the switch from bulk antiferromagnetism to superlattice ferromagnetism. The antiferromagnetic crystal has a unit cell

TABLE II. Components of the magnetization (in μ_B) projected on the outer and inner plane Mn in the $(\text{BaTiO}_3)_{4.5}(\text{LaMnO}_3)_{4.5}$ superlattice at two different energy minima. Discrepancies between the M_y of equivalent atoms in the two outer planes, of $0.002\mu_B$ or less, have been averaged out. Note that the magnitude of M on every atom is independent of energy minimum.

| | M_x | M_y | M_z | M_x | M_y | M_z |
|---------------------|-------|-------|--------------------|-------|-------|-------|
| Out1 | 0.014 | 4.301 | 0.012 | 0.618 | 4.255 | 0.111 |
| Out2 | 0.007 | 4.675 | 0.009 | 0.818 | 4.601 | 0.116 |
| In1 | 0.014 | 4.254 | 0.012 | 0.747 | 4.187 | 0.099 |
| In2 | 0.010 | 4.258 | 0.012 | 0.616 | 4.212 | 0.102 |
| $E=-727.1502368$ eV | | | $E=-727.149665$ eV | | | |

twice as large as the ferromagnetic crystal, resulting in extra energy gaps and therefore a lowering of the band energy, but at a cost of reducing the magnitude of the exchange energy. Which of these contributions to the energy dominates depends on the lattice constants. We performed a bulk LaMnO_3 calculation using superlattice lattice parameters (a and b from Table I and $c=7.8928$ Å, the distance between the second and fourth LaO planes in the superlattice) and found it to be ferromagnetic. This alone explains why the superlattice is ferromagnetic. We found that the magnetization was larger on the interface planes than on the interior planes and, experimentally,²⁰ as previously mentioned, when the interfaces are dense, the Curie temperature is significantly higher than that of the bulk crystal, while when they are far apart, it is much lower than the bulk Curie temperature. Thus, interfaces increase the magnetization and may favor ferromagnetism over antiferromagnetism.

There are four Mn planes in the superlattice, each containing two inequivalent Mn atoms. The two inner planes are equivalent, as are the outer planes. Two sets of the components of the magnetization of these atoms are listed in Table II, which were obtained by including the spin-orbit interaction and allowing for possible noncollinearity. The ground state calculation was initiated with a ferromagnetic spin distribution. The metastable state³⁷ calculation was initiated with a random spin distribution [except for the identity of the inner (outer) plane spins]. As shown in Table II, all the ground state spins are polarized essentially along the \mathbf{b} lattice vector,³⁸ whereas the metastable state Mn spins are not quite collinear and point in a direction of no particular symmetry. Because they are identical on the two outer (inner) planes, we believe that the small values of M_x and M_z in the ground state are real and not an artifact of the calculation. Note that the metastable and ground state spin *magnitudes* are identical. The two inner plane atom magnetizations are very slightly larger than those calculated for bulk LaMnO_3 , whereas one of the outer two is slightly larger and one is significantly larger. This can be understood with an examination of the outer plane Mn bond lengths to their six neighboring oxygen atoms, which are listed in Table III. Since covalent bonds are not magnetic and longer bond lengths imply less covalency, the Mn with the longer bonds are more magnetic.

TABLE III. Comparison of the six nearest neighbor oxygen distances (in Å) to the Mn in LaMnO_3 and to the outer plane Mn in the $(\text{BaTiO}_3)_{4.5}(\text{LaMnO}_3)_{4.5}$ superlattice.

| Bulk | Mn1 | Mn2 |
|-------|-------|-------|
| 1.938 | 2.006 | 2.146 |
| 1.938 | 2.009 | 2.147 |
| 2.041 | 2.028 | 2.153 |
| 2.041 | 2.021 | 2.155 |
| 2.078 | 2.115 | 2.078 |
| 2.078 | 2.186 | 2.155 |

The calculated superlattice ferroelectric polarization is $13.079 \mu\text{C}/\text{cm}^2$. It lies along the direction in which the BaTiO_3 is stretched by the LaMnO_3 , i.e., along the same \mathbf{b} lattice vector as the magnetization. It is about half as large as that calculated for bulk BaTiO_3 , which was to be expected because the BaTiO_3 constitutes about half of the volume of the superlattice. It can be compared with the experimental value of $12 \mu\text{C}/\text{cm}^2$ obtained for the $\text{La}_{0.7}\text{Ca}_{0.3}\text{MnO}_3/\text{BaTiO}_3$ superlattice.²⁰ The Berry phase polarization is uncertain to within some number of quanta,³⁵ $\Delta\mathbf{P}=e\mathbf{b}/\Omega=7.900 \mu\text{C}/\text{cm}^2$, where e is the electronic charge, \mathbf{b} is the lattice vector, and Ω is the unit cell volume. To prove that no quanta should be added or subtracted from our result, we have calculated the polarization with the atoms displaced 75%, 50%, 25%, 10%, and 0% from their centrosymmetric positions and plotted the curve in Fig. 1. No other smooth curve can be drawn by subtracting any number of quanta from each point.³⁹

It has been suggested⁴⁰ that in the case of a superlattice consisting of a ferroelectric and a metal where the Berry phase method is not applicable, one can estimate the polarization from the ratio of superlattice displacements away

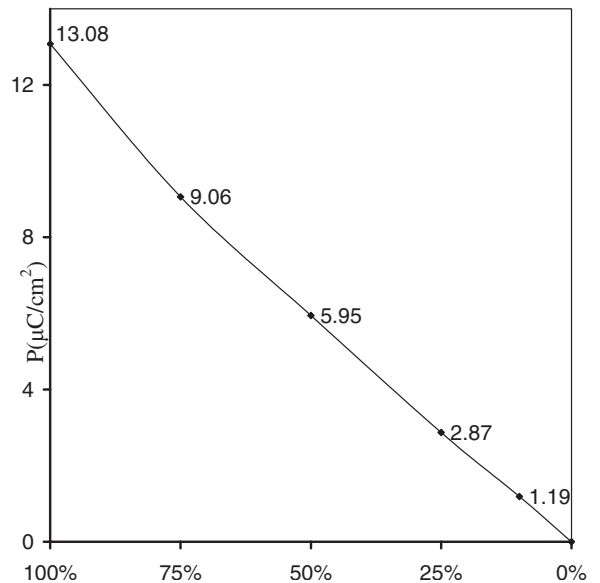


FIG. 1. Polarization as a function of the percentage of the atomic displacements from their centrosymmetric positions toward their equilibrium positions.

TABLE IV. Berry phase polarizations (in $\mu\text{C}/\text{cm}^2$), unit cell volumes, and the averaged displacements of the Ti and O atoms in the $(\text{BaTiO}_3)_{4.5}(\text{LaMnO}_3)_{4.5}$ superlattice compared with bulk BaTiO_3 and the superlattice polarizations P_{Ti} and P_{O} estimated therefrom.

| | P | V (\AA^3) | ΔTi (\AA) | P_{Ti} | ΔO (\AA) | P_{O} |
|-----------|--------|---------------------------|---------------------------------------|-----------------|--------------------------------------|----------------|
| Bulk | 27.05 | 61.18861 | 0.024799 | | 0.029807 | |
| Supercell | 13.088 | 1158.059 | 0.020692 | 11.925 | 0.033566 | 16.094 |

from centrosymmetry to those in the bulk crystal. It seemed that this would be a good opportunity to test that approximation. We have $P_{\text{est}} = (D/d)(v/V)P_{\text{bulk}}$, where D (d) is the sum of the superlattice (bulk crystal) displacements for either Ti or O and V (v) is the superlattice (bulk crystal) unit cell volume. In the first column of Table IV, P_{bulk} and $P_{\text{supercell}}$ are listed. The remaining columns list the average (for ease of comparison) Ti and O displacements and P_{Ti} and P_{O} , the polarizations estimated therefrom. We see that $P_{\text{Ti}} = 0.911 P_{\text{supercell}}$ and $P_{\text{O}} = 1.230 P_{\text{supercell}}$ and that their average, $P_{\text{est}} = 1.070 P_{\text{supercell}}$ is a reasonable approximation. The outer plane Mn are displaced relative to the inner plane Mn by 0.33 and 0.28 \AA in the opposite direction to the Ti displace-

ment along the \mathbf{b} lattice vector. This is not included in the approximate calculation and could account for its discrepancy with respect to the Berry phase calculation.

In summary, we have calculated the properties of antiferromagnetic LaMnO_3 and ferroelectric BaTiO_3 and of a $(\text{LaMnO}_3)_{4.5}(\text{BaTiO}_3)_{4.5}$ superlattice with LaO/TiO_2 interfaces. We found that the superlattice is ferromagnetic and showed that this was a consequence of the difference between bulk crystal and superlattice lattice constants, although interface effects, which strengthened the magnetization, might also favor ferromagnetism. The electric polarization density was calculated using the Berry phase method and found, as expected, to be about half that of BaTiO_3 . Both the magnetic and electric polarizations were found to point along the in-plane \mathbf{b} lattice vector. An approximate method, which must be used when the system is metallic, was found to be in fairly good agreement with the Berry phase result.

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- ³⁹It is interesting to note that a very smooth curve can be drawn by subtracting 4, 3, 2, 1, and 0 quanta from the values with 100%, 75%, 50%, 25%, and 0% displacements, respectively, but there is no way to make the point at 10% displacement lie on that curve.
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